Differentiability

Piecewise functions may or may not be differentiable on their domains.

To be differentiable at a point \( x = c \), the function must be continuous, and we will then see if it is differentiable.

Let’s consider some piecewise functions first.

Let \( f(x) = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases} \)

First we will check to prove continuity at \( x = 0 \)

\[
\begin{align*}
    f(0) &= 0 \\
    \lim_{x \to 0^-} f(x) &= 0 \\
    \lim_{x \to 0^+} f(x) &= 0
\end{align*}
\]

Now we will consider differentiability at \( x = 0 \)

\[
\begin{align*}
    \lim_{x \to 0^-} f'(x) &= -1 \\
    \lim_{x \to 0^+} f'(x) &= 1
\end{align*}
\]

\( f(x) \) is continuous but \textbf{not} \textit{diff. at } x = 0
Now let’s try:

\[ g(x) = \begin{cases} 
8x - 3, & x \leq 1 \\
4x^2 + 5, & x > 1 
\end{cases} \]

Is \( g(x) \) continuous AND differentiable at \( x = 1 \)?

\[ g(1) = 5 \]
\[ \lim_{x \to 1^-} g(x) = 5 \]
\[ \lim_{x \to 1^+} g(x) = 9 \]

\( g(x) \) is not continuous and differentiable at \( x = 1 \).

Is \( h(x) \) continuous and differentiable at \( x = 3 \)?

\[ h(x) = \begin{cases} 
x^2 - 4x + 8, & x \leq 3 \\
2x - 1, & x > 3 
\end{cases} \]

\[ h(3) = 5 \]
\[ \lim_{x \to 3^-} h(x) = 5 \]
\[ \lim_{x \to 3^+} h(x) = 5 \]

\[ \lim_{x \to 3^-} h'(x) = 2 \]
\[ \lim_{x \to 3^+} h'(x) = 2 \]

\( h'(x) = \begin{cases} 
2x - 4, & x \leq 3 \\
2, & x > 3 
\end{cases} \)

\( h(3) = \lim_{x \to 3^-} h'(x) = 2 \)
Now let’s try something trickier:

\[ f(x) = \begin{cases} 
3x^2 + 4x, & x \leq 1 \\
2x^3 + bx + c, & x > 1 
\end{cases} \]

If \[ f'(x) = \begin{cases} 
6x + 4, & x \leq 1 \\
6x^2 + b, & x > 1 
\end{cases} \]

Find \( b \) and \( c \) so that \( f(x) \) is differentiable at \( x = 1 \)

Let’s work on continuity first.

\[ f(1) = 7 \]
\[ \lim_{x \to 1^-} f(x) = 7 \]
\[ \lim_{x \to 1^+} f(x) = 2 + b + c \]

Now work on differentiability

\[ \lim_{x \to 1^-} f'(x) = 10 \]
\[ \lim_{x \to 1^+} f'(x) = 6 + b \]

\( 7 = 2 + b + c \)

\( 10 = 6 + b \)

\( 4 = b \)

\( 1 = c \)
Let \( f(x) = \begin{cases} \ ax^2 + 10, & x < 2 \\ x^2 - 6x + b, & x \geq 2 \end{cases} \)

Find the values of \( a \) and \( b \) such that \( f(x) \) is differentiable

\[
\lim_{x \to 2^{-}} f(x) = 4a + 10 \\
\lim_{x \to 2^{+}} f(x) = -8 + b \\
\lim_{x \to 2^{-}} f'(x) = 4a \\
\lim_{x \to 2^{+}} f'(x) = -2
\]

\[
4a + 10 = -8 + b \\
b = 16 \\
4a = -2 \\
a = -\frac{1}{2}
\]

Homework: page 118 #113 and #114 [show ALL steps with GOOD notation]